DISCRETE MORSE FLOW FOR RICCI FLOW AND POROUS MEDIA EQUATION

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Abstract. In this paper, we study the discrete Morse flow for the Ricci flow on football, which is the 2-sphere with removed north and south poles and with the metric g_0 of constant scalar curvature, and and for Porous media equation on a bounded regular domain in the plane. We show that with a suitable assumption about $g(0)$ we have a weak approximated discrete Morse flow for the approximated Ricci flow and Porous media equation on any time intervals.

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1. Introduction

There are relative few results about computational models for the Ricci flow in two dimensions. The purpose of this paper is to try this area by giving some approximated computational models, namely the discrete Morse flow for the 2-d Ricci flow. We shall first consider the Porous-media equation on a bounded regular domain in the plane. As is well-known that the limiting equation of the Porous-Media equation is the Ricci flow on the domain. Then we may consider this discrete flow as the approximated computational scheme for the Ricci flow. We also consider two different modelings of the Ricci flow on the singular surface, the American football. Our method can also be worked out in a similar way in the regular spherical surfaces and other surfaces.

Since the geometric and analytic parts of singular surfaces are not well-known, we now recall the geometry of a special singular surface, namely, the American football. Let $S = S_0$ be the sphere with north and south poles removed and with the metric

$$g_0 = dr^2 + (\alpha \sin r)^2 d\theta^2,$$

where $\alpha \in (0, 1)$, $0 \le r \le \pi$ such that $r = 0$ corresponding the north pole, and $0 \le \theta \le 2\pi$. With this metric, we can see that S_0 has two singularities of equal angle $2\pi\alpha$ at the north and south poles. By a direct computation we know that the scalar curvature of the metric g_0 is equal to 2. Recall here that the Laplacian operator of g_0 on function is

$$\Delta_0 = \partial_r^2 + \frac{\cos r}{\sin r}\partial_r + \frac{1}{(\alpha \sin r)^2}\partial_\theta^2$$

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and the area element is

$$dA = \alpha \sin r dr d\theta.$$

The area of (S, g_0) is $|A| = 4\pi\alpha$. The aim of this note is to study the normalized Ricci flow on (S, g_0):

$$\partial_t g = (\rho - R) g$$

where $g = g(t) = e^{2u(t)} g_0$, $R = R(g)$ is the scalar curvature of the metric g, and ρ is some real constant (and we may choose it such that the area of the flow $g(t)$ is constant). We may write the evolution equation of the Ricci flow as

$$e^u \partial_t e^u = \frac{1}{2}\rho e^{2u} + \Delta u - 1,$$

where Δ is the Laplacian operator of the metric g_0. The standard example is that for the football metric $g_0 = dr^2 + (\alpha \sin r)^2 d\theta^2$ $(0 < \alpha < 1)$ the (un-normalized) Ricci flow is $g(t) = (1 - 2t) g_0$ for $t < \frac{1}{2}$, which extinct at $t = \frac{1}{2}$.

For the construction of the discrete Morse flow for the 2-d Ricci flow, we shall rely on some variational structure related. Hence we need some analytical part of the related variational functionals. Here we recall one important consequence of a Moser type inequality proven by W.Chen and C.Li [6]. For any $\phi \in H^1(S)$,

$$\int_S e^{\phi} dA \leq C_2 exp\{\frac{1}{16\pi\alpha}\int_S |\nabla\phi|^2 dA + \frac{1}{|S|}\int \phi dA\} \tag{1}$$

and the functional

$$\phi \in H^1(S) \to \int_S e^{\phi} dA$$

is continuous with respect to the weak convergence in H^1. For our purpose, we reformulate the inequality above as For any $u \in H^1(S)$,

$$⨍_S e^{2u} dA \leq C_2 exp\{⨍_S |\nabla u|^2 dA + ⨍_S 2u dA\} \tag{2}$$

with $⨍ u dA = \frac{1}{|S|}\int u dA$. We shall use this fact to get a weak solution to the Ricci flow on (S, g_0).

The precise results will be stated and proved in the following three sections. Similar results for Yamabe flow and higher dimensional Porous-media equations are also true. The plan of this paper is below. In section 2, we study the existence of a weak solution to the porous media equation in a bounded regular domain. In section 3, we introduce the method of the discrete Morse flow method to an approximated ricci flow in the singular surface with symmetry the initial data. We study a perturbated Ricci flow model in the last section.

2. discrete Morse flow for the Porous-media equation

The existence of the weak solution to the Porous-media equation can be done by using the Galerkin method [2]. Other related methods can be found in [8]. Here we propose a new method, which is the adapted discrete Morse flow method [13].

To make our idea more clear, we start from the discrete Morse flow for the Porous-Media equation on the bounded regular domain Ω in the plane R^2. In some sense our domain be be any one with the Sobolev imbedding theorem holds true.

Assume that $m > 0$. Given an initial regular data u_0 and any $T > 0$. We consider the porous media equation

$$\partial_t u = \Delta u^m, \quad \Omega \times [0, T], \tag{3}$$

with the initial data $u|_{t=0} = u_0$ and with the boundary condition $u(t) = u_0$ on $\partial\Omega \times t$.

This equation has a very close relation with the Ricci flow. In fact, by taking the limit $m \to 0$, the limiting equation of (3) is

$$\partial_t u = \Delta \log u, \tag{4}$$

which is the Ricci flow on the plane R^2.

To introduce the discrete Morse flow for (3) we set $v = u^m$ and let $\alpha = 1/m$. Then (3) is induced into

$$\partial_t v^\alpha = \Delta v \tag{5}$$

with the initial data $v = v_0 = u_0^\alpha$ at $t = 0$. We assume that $v_0 \in H^1(\Omega)$. The key idea is that we set $\alpha = 2\beta - 1$ with $\beta > 1/2$ and make

$$\partial_t v^\alpha = C_\beta v^{\beta-1} \partial_t v^\beta,$$

where $C_\beta > 0$ such that $\frac{\beta}{2\beta-1} C_\beta = 1$.

In below, we assume that $\beta > 1$ and make the notation that $v^\beta = |v|^\beta$ for any v. For any $N > 1$ be a large integer and for any $T > 0$,let

$$h = T/N, \quad t_n = nh, \quad n = 0, 1, 2, ..., N.$$

Assume that we have constructed $v_j \in H^1(S)$, $0 \leq j \leq n-1$ and u_{n-1} is a minimizer of the functional

$$I_{n-1}(u) = \frac{C_\beta}{2h} \int_\Omega |v^\beta - v_{n-2}^\beta|^2 dx + \frac{1}{2} \int_\Omega |\nabla v|^2 dx$$

on $H := \{v \in H^1(\Omega); v - v_0 \in H_0^1(\Omega)\}$. Define

$$I_n(u) = \frac{C_\beta}{2h} \int_\Omega |v^\beta - v_{n-1}^\beta|^2 dx + \frac{1}{2} \int_\Omega |\nabla v|^2 dx$$

on H. It is clear that the infimum is finite and by using the Poincare inequality to $v - v_0$, any minimizing sequence is bounded in H. We remark here that we may choose the functions in the minimizing sequence non-negative since our functional is even functional [1]. By the direct method, we know that I_n has an unique minimizer u_n in H, which satisfies that

$$\frac{\beta C_\beta}{h} (v^\beta - v_{n-1}^\beta) v^{\beta-1} = \Delta v$$

with the uniform energy bound

$$\frac{C_\beta}{2h} \int_\Omega |v_n^\beta - v_{n-1}^\beta|^2 dx + \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx \leq \frac{1}{2} \int_\Omega |\nabla v_{n-1}|^2 dx \leq C.$$

We inductively define $v_N(t) \in H^1$ for $t \in [-h, T]$ such that for $n = 1, ..., N$,

$$v_N(t) = v_n,$$

on $[t_{n-1}, t_n]$ and $v_N(t) = v_0$ on $[-h, 0]$. We define, for $1 \le n \le N$,

$$\partial_t v_N^\beta(t) = \frac{1}{h}(v_n^\beta - u_{n-1}^\beta), \quad t \in [t_{n-1}, t_n],$$

and $\tilde{v}_N(t) = v_N(t-h)$. Taking the convergent subsequence in the weak star topology in H, we know that the limit $v \in H$ satisfies

$$C_\beta v^{\beta-1}\partial_t v^\beta = \Delta v$$

in the distributional sense. The latter is the weak form of the equation (5).

Then we have proven the following result.

Theorem 1. *Assume that Ω is a regular domain in the plane R^2. Given any $T > 0$ and $m > 0$. Assume that the initial data $v_0 = u_0^{1/m} \in H^1(\Omega)$. Let $\alpha = 1/m = 2\beta - 1$ with $\beta > 1$. Then there is at least one weak solution $v \in L^\infty H^1(\Omega)$ to the porous media equation (5).*

Recall here that a mapping $v : [0, T] \to H^1(\Omega)$ is said *a weak solution* to (5) if $v \in\in L^\infty H^1(\Omega)$ satisfies the evolution equation (5) in the distributional sense and $\lim_{t\to 0} u(t) = u_0$ in $L^2(\Omega)$ and $u(t, x) = u_0(x)$ for $x \in \partial\Omega$ in the trace sense. The latter will be simply said that $u(t)$ has the initial data u_0.

3. H^1 weak solution to Ricci flow on (S, g_0) with symmetry

We now choose $\rho = \frac{2}{⨍_S e^{2u} dA}$ and fix any positive constant $T > 0$. Fix any $u_0 \in H^1$ with $u(x) = u(-x)$, which is symmetric about $r = \pi/2$. Then the Ricci flow equation under consideration is

$$e^u \partial_t e^u = \Delta u - 1 + \frac{e^{2u}}{⨍ e^{2u} dA}, \quad in \ S \times (0, T] \tag{6}$$

with the initial data $u(0) = u_0$. The symmetry condition is used to get the compactness required for the minimization process below. We now introduce a new concept of H^1 weak solution to the Ricci flow (6).

Definition 2. *We say $u(t) \in H^1$, $t \in [0, T]$ is a weak solution to the Ricci flow (6) with the initial data u_0 if it satisfies (6) in the sense of distribution and with the bounds*

$$\sup_{t\in[0,T]} ⨍_S |\nabla u|^2 + 2u dA - \log ⨍_S e^{2u} dA \le ⨍_S |\nabla u_0|^2 + 2u_0 dA - \log ⨍_S e^{2u_0} dA.$$

and

$$\int_{S\times[0,T]} |\partial_t e^{u(t)}|^2 \le \frac{1}{2}\int_S |\nabla u_0|^2 + 2u_0 dA - \frac{1}{2}\log ⨍_S e^{2u_0} dA.$$

In below, we may assume that u_0 is smooth (i.e., $u_0 \in C^2$) and normalize it such that the average $\bar{u}_0 = ⨍_S u = \frac{1}{|S|}\int_S u_0 dv_{g_0}$, otherwise, we choose an smoothly approximation of u_0 and passing to subsequence limit.

For any $N > 1$ be a large integer and for any $T > 0$,let

$$h = T/N, \quad t_n = nh, \quad n = 0, 1, 2, ..., N.$$

Assume that we have constructed $u_j \in H^1(S)$, $0 \leq j \leq n-1$ and u_{n-1} is a minimizer of the functional

$$J_{n-1}(u) = \frac{1}{2h} \fint_S |e^u - e^{u_{n-2}}|^2 dA + \frac{1}{2} \fint_S |\nabla u|^2 dA - \frac{1}{2} \log \fint_S e^{2u} dA$$

over $\mathbf{A} = \{u \in H^1; \bar{u} = 0\}$ and with the uniform bound

$$\fint_S |\nabla u_{n-1}|^2 dA - \log \fint_S e^{2u} dA \leq C.$$

We want to get another u_n with same bound by using the discrete Morse method due to Rothe . Define the functional

$$J_n(u) = \frac{1}{2h} \fint_S |e^u - e^{u_{n-1}}|^2 dA + \frac{1}{2} \fint_S |\nabla u|^2 dA - \frac{1}{2} \log \fint_S e^{2u} dA$$

in $\mathbf{A}$. If u_n is a minimizer of the functional on $\mathbf{A}$, then it is clear that its Euler-Lagrange equation of this functional is

$$e^{u_n} \frac{1}{h}(e^{u_n} - e^{u_{n-1}}) = \Delta u_n - \lambda_n + \frac{e^{2u_n}}{\fint_S e^{2u_n}}, \quad on \ S, \tag{7}$$

where

$$\lambda_n = 1 - \fint_S e^{u_n} \frac{1}{h}(e^{u_n} - e^{u_{n-1}}),$$

and the minimizer satisfies the following estimate

$$J_n(u_n) \leq J_n(u_{n-1}) = \frac{1}{2} \int_S |\nabla u_{n-1}|^2 dA - \frac{1}{2} \log \fint_S e^{2u_{n-1}} dA. \tag{8}$$

We want to minimize this functional on $\mathbf{A}$. First we need to know that the infimum is finite. This follows from the fact that after using the inequality (1), the leading term in the functional J_n is $\frac{1}{2h} \int_S |e^u - e^{u_{n-1}}|^2 dA$.

To get the minimizer, we only need to show that the minimizing sequence is bounded in $\mathbf{A}$. Right from the the relation

$$\inf_{H^1(S)} J_n(u) \leq J_n(u_{n-1}) \leq C,$$

we know that the minimizing sequence ($u = u_{nk}$) satisfies that

$$\frac{1}{2h} \fint_S |e^u - e^{u_{n-1}}|^2 dA + \frac{1}{2} \fint_S |\nabla u|^2 dA - \frac{1}{2} \log \fint_S e^{2u} dA \leq C.$$

Using Chen-Li's Moser type inequality we know that the minimizing sequence is uniformly bounded in H^1. By this we can pass to subsequence to get a weakly convergent subsequence with its limit u_n as the minimizer.

Once this is done, we inductively define $u_N(t) \in H^1$ for $t \in [-h, T]$ such that for $n = 1, ..., N$,

$$u_N(t) = u_n, \quad \lambda_N(t) = \lambda_n,$$

on $[t_{n-1}, t_n]$ and $u_N(t) = u_0$ on $[-h, 0]$. We define, for $1 \leq n \leq N$,

$$\partial_t e^{u_N(t)} = \frac{1}{h}(e^{u_n} - e^{u_{n-1}}), \quad t \in [t_{n-1}, t_n],$$

and $\tilde{u}_N(t) = u_N(t-h)$. The estimate (8) gives us that

$$\sup_{t\in[0,T]} \fint_S |\nabla u_n|^2 dA - \frac{1}{2}\log \fint_S e^{2u_n} dA \leq \int_S |\nabla u_0|^2 dA - \frac{1}{2}\log \fint_S e^{2u_0} dA.$$

and

$$\int_{S\times[0,T]} |\partial_t e^{u_N(t)}|^2 \leq \frac{1}{2}\fint_S |\nabla u_0|^2 dA - \frac{1}{2}\log \fint_S e^{2u_0} dA + C_0.$$

The uniform upper bound of $\int_S |\nabla u_N|^2 dA$ follows from the uniform bound of $\int_S e^{u_N}$ and the assumption that $u(x) = u(-x)$ where the reflection is about $r = \pi/2$. With the latter symmetry assumption we have the better inequality in the sense of Moser that

$$\fint_S e^{2u} dA \leq C_2 exp\{\frac{1}{2}\fint_S |\nabla u|^2 dA + \fint_S 2u dA\}$$

which is proved by W.Chen [5] (see theorem II there). It is here that the symmetry condition plays the role.

Note that $\{u_N\}$ is bounded in $L_t^\infty H^1$ and $|\partial_t e^{u_N}|$ is bounded in $L^2([0,t]\times S)$ for any $t > 0$. Then we may pass to weakly convergent subsequence of $u_N(t)$ in (7) to get the weak limit $u(t) \in L_t^\infty H^1(S) \cap L^2([0,t]\times S)$ such that

$$e^u \partial_t e^u = \Delta u - \lambda(t) + \frac{e^{2u}}{\fint_S e^{2u} dA}, \tag{9}$$

where

$$\lambda(t) = \lim \lambda_N(t) = 1 - \fint_S e^u \partial_t e^u.$$

The latter is called the relaxation parameter of the ricci flow and the open problem is to prove that it decays to zero as time going to infinity.

Hence we have proven the following assertion.

Theorem 3. *For any positive constant $T > 0$ and any $u_0 \in H^1$ with $u(x) = u(-x)$ with respect to $r = \pi/2$, there exists at least one weak solution to (6) on $S \times [0,T]$ in the sense of the weak form (9).*

Our argument above also work in the case of the standard sphere. It may be used to define the weak solution to the Kaehler-Ricci flow. It is an open question if we have stability result or uniqueness result for the weak solution above. We remark that it is possible to get a little more regularity of the weak solution for smooth initial data u_0. For higher dimensions, except the concept of Ricci flow with surgery, it is not clear if one can define the weak Ricci flow.

4. Another discrete Morse flow

Fix $\lambda \in (0,1)$. This parameter will play the regularization role in the discrete Ricci flow below. We study the following modified Ricci flow

$$e^u \partial_t e^u = \Delta u - \lambda + \frac{e^{2u}}{\fint e^{2u} dA}, \quad in \ \ S\times(0,T] \tag{10}$$

with the initial data $u(0) = u_0 \in H^1$. In this case, the variation structure related to the right side of (10) is

$$I(u) = \frac{1}{2}\fint_S (|\nabla u|^2 + 2\lambda u) dA - \frac{1}{2}\log \fint_S e^{2u} dA.$$

The advantage of this function is that the inequality (2) implies that

$$I(u) \geq 2)(\lambda - 1) \fint_S u dA \tag{11}$$

and

$$I(u) \geq \frac{1-\lambda}{2} \fint |\nabla u|^2 dA - \frac{1-\lambda}{2} \log \fint_S e^{2u} dA. \tag{12}$$

These two facts will help us to make the infimum of the functional $\hat{J}$ (see below) in H^1 be finite.

As in last section, for any $N > 1$ be a large integer and for any $T > 0$,let

$$h = T/N, \quad t_n = nh, \quad n = 0, 1, 2, ..., N.$$

Assume that we have constructed $\hat{u}_j \in H^1(S)$, $0 \leq j \leq n-1$ and $\hat{u}_{n-1}$ is a minimizer of the functional

$$\hat{J}_{n-1}(u) = \frac{1}{2h}\fint_S |e^u - e^{\hat{u}_{n-2}}|^2 dA + \frac{1}{2}\fint_S (|\nabla u|^2 + 2\lambda u) dA - \frac{1}{2}\log \fint_S e^{2u} dA$$

over H^1 and with the uniform bound

$$\fint_S (|\nabla u_{n-1}|^2 + 2\lambda u) dA - \log \fint_S e^{2u} dA \leq C.$$

Let

$$\hat{J}(u) = \frac{1}{2h}\fint_S |e^u - e^{\hat{u}_{n-1}}|^2 dA + I(u)$$

on H^1. Because of the inequality (2) we can easily show as before that the minimizing sequence of $\hat{J}$ is bounded in H^1. In fact, if the infimum is ∞, then we may have sequence $w_j \in H^1$ such that $\hat{J}(w_j) \to -\infty$. By (12) or (11) we know that $\int_S e^{2w_j} \to \infty$ or $\fint_S w_j \to \infty$, which is impossible since in this case the leading term is

$$\frac{1}{h}\fint_S |e^{w_j} - e^{u_{n-1}}|^2.$$

Hence the infimum is finite. By this we can pass to subsequence to get a weakly convergent subsequence with its limit $\hat{u}_n$ as the minimizer of $\hat{J}$. The Euler-lagrange equation of $\hat{J}$ is

$$e^{\hat{u}_n}\frac{1}{h}(\hat{e}^{u_n} - e^{\hat{u}_{n-1}}) = \Delta \hat{u}_n - \lambda + \frac{e^{2\hat{u}_n}}{\fint_S e^{2\hat{u}_n}}, \quad on \;\; S. \tag{13}$$

By the standard elliptic regularity theory we know that $\hat{u}_n$ is smooth.

Then we inductively define $\hat{u}_N(t) \in H^1$ for $t \in [-h, T]$ such that for $n = 1, ..., N$,

$$\hat{u}_N(t) = \hat{u}_n,$$

on $[t_{n-1}, t_n]$ and $\hat{u}_N(t) = u_0$ on $[-h, 0]$. We define, for $1 \le n \le N$,

$$\partial_t e^{\hat{u}_N(t)} = \frac{1}{h}(e^{\hat{u}_n} - e^{\hat{u}_{n-1}}), \quad t \in [t_{n-1}, t_n]$$

the minimizing property of $\hat{u}_n$ gives us the desired energy bound to obtain a convergent subsequence $\hat{u}_N$ and the limit $\hat{u} \in L^\infty_t H^1(S) \cap L^2([0,t] \times S)$ such that

$$e^u \partial_t e^u = \Delta u - \lambda + \frac{e^{2u}}{\fint_S e^{2u} dA}. \tag{14}$$

Hence we have the following result.

Theorem 4. *For any positive constant $T > 0$ and any $u_0 \in H^1$, there exists at least one weak solution to (10) on $S \times [0, T]$ in the sense of the weak form (14).*

Here the weak solution of (10) is defined similar to the definition 2.

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